

## THE HEISENBERG GROUP ACTS ON A STRICTLY CONVEX DOMAIN.

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Every linear group acts by isometries on some properly convex domain in real projective space. This follows from the fact that action of  $\mathrm{SL}(n, \mathbb{R})$  on the space of quadratic form in  $n$  variables preserves the properly convex cone consisting of positive definite forms. If  $\Gamma$  is the holonomy of a properly convex orbifold of finite volume then every virtually nilpotent group is virtually abelian, moreover every unipotent element is conjugate into  $\mathrm{PO}(n, 1)$ . A reference for all this is [1]. This paper gives the first example of a unipotent group that is not virtually abelian and preserves a strictly convex domain. It answers a question asked by Misha Kapovich.

The *Heisenberg group* is the subgroup  $H \subset \mathrm{SL}(3, \mathbb{R})$  of unipotent upper-triangular matrices. Define  $\theta : H \rightarrow \mathrm{SL}(10, \mathbb{R})$  and  $G = \theta(H)$  where

$$\theta \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2a & 2c & a & a^2/2 & a^3/6 & b & 2a^2 + b^2/2 & b^3/6 + 2ac & (a^4 + b^4)/24 + c^2 \\ 0 & 1 & b & 0 & 0 & 0 & 0 & 2a & ab + c & bc \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & a & c \\ 0 & 0 & 0 & 1 & a & a^2/2 & 0 & 0 & 0 & a^3/6 \\ 0 & 0 & 0 & 0 & 1 & a & 0 & 0 & 0 & a^2/2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & b & b^2/2 & b^3/6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & b & b^2/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & b \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

It is clear that  $\theta$  is injective and easy to check that it is a homomorphism. Since the center of  $H$  is  $Z \cong \mathbb{R}$  and  $H/Z \cong \mathbb{R}^2$  it is also easy to check that every non-trivial element of  $G$  has a unique largest Jordan block, and that this block has odd size. It easily follows that each element of  $G$  preserves some properly convex domain depending on that element, cf the discussion of parabolics in (2.9) of [1].

**Theorem 0.1.** *There is a strictly convex domain  $\Omega \subset \mathbb{RP}^9$  that is preserved by  $G$ . This is an effective action of the Heisenberg group on  $\Omega$  by parabolic isometries that are unipotent.*

*Proof.* The group  $G$  acts affinely on the affine patch  $[x_1 : x_2 : x_3 : x_4 : x_5 : x_6 : x_7 : x_8 : x_9 : 1]$  that we identify with  $\mathbb{R}^9$ . Let  $p \in \mathbb{R}^9$  be the origin. Then  $G \cdot p$  is

$$((a^4 + b^4)/24 + c^2, bc, c, a^3/6, a^2/2, a, b^3/6, b^2/2, b)$$

This orbit is an algebraic embedding  $\mathbb{R}^3 \hookrightarrow \mathbb{R}^9$  which limits on the single point

$$q = [1 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0] \in \mathbb{RP}^9$$

in the hyperplane at infinity,  $P_\infty$ . This follows from the fact that  $(a^4 + b^4)/24 + c^2$  dominates all the other entries whenever at least one of  $|a|, |b|, |c|$  is large.

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Let  $S \subset \mathbb{R}^9$  be this orbit. Choose 10 random points on  $S \subset \mathbb{RP}^9$  and compute the determinant,  $d$ , of the corresponding 10 vectors in  $\mathbb{R}^{10}$ . Then  $d \neq 0$  therefore the interior  $\Omega^+ \subset \mathbb{R}^9$  of the convex hull of  $S$  has dimension 9.

Moreover the closure  $\Omega'$  of  $\Omega^+$  in  $\mathbb{RP}^9$  is disjoint from the closure of the affine hyperplane  $x_1 = -1$ , hence  $\Omega^+$  is properly convex. Since  $\Omega' \cap P_\infty = q$  and  $G$  preserves  $q$  and  $P_\infty$  and  $G$  is unipotent, it follows from (5.8) in [1] that  $G$  preserves some strictly convex domain  $\Omega \subset \Omega'$ .  $\square$

**Corollary 0.2.** *There is a strictly convex real projective manifold  $\Omega/\Gamma$  of dimension 9 with nilpotent fundamental group  $\Gamma \cong \langle \alpha, \beta : [\alpha, [\alpha, \beta]], [\beta, [\alpha, \beta]] \rangle$  that is not virtually abelian. Moreover  $\Gamma$  is unipotent.*

*Proof.* If  $\Gamma$  is a lattice in  $G$  then  $\Omega/\Gamma$  is a strictly convex manifold with unipotent holonomy and  $\Gamma$  is nilpotent but not virtually abelian.  $\square$

The genesis of this example is as follows. The image of  $H$  in  $\mathrm{SL}(6, \mathbb{R})$  under the irreducible representation  $\mathrm{SL}(3, \mathbb{R}) \rightarrow \mathrm{SL}(6, \mathbb{R})$  is

$$\begin{pmatrix} 1 & 2a & a^2 & 2c & 2ac & c^2 \\ 0 & 1 & a & b & ab+c & bc \\ 0 & 0 & 1 & 0 & 2b & b^2 \\ 0 & 0 & 0 & 1 & a & c \\ 0 & 0 & 0 & 0 & 1 & b \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and preserves the properly convex domain  $Q \subset \mathbb{RP}^5$  that is the projectivization of the space of positive definite quadratic forms on  $\mathbb{R}^3$ .

The boundary of the closure of  $Q$  consists of semi-definite forms and contains flats, so  $Q$  is *not strictly convex*. Let  $A, B, C \in \mathrm{SL}(6, \mathbb{R})$  be the elements corresponding to one of  $a, b, c$  being 1 and the others 0. Each of  $A, B, C$  has a parabolic fixed point in  $\partial Q$  corresponding to a rank 1 quadratic form. Every point in  $Q$  converges to this parabolic fixed point under iteration by the given group element. The fixed point for  $A$  and  $B$  are distinct and lie in a flat in  $\partial Q$ .

The idea is to increase the dimension of the representation and use the extra dimensions to add parabolic blocks of size 5 onto  $A$  (row 1 and rows 7-10) and onto  $B$  (row 1 and rows 11-14) that commute and the parabolic fixed point of each block is the rank-1 form that is a fixed point of  $C$ . This gives a 14-dimensional representation of  $H$ :

$$\begin{pmatrix} 1 & 2a & a^2 & 2c & 2ac & c^2 & a & a^2/2 & a^3/6 & a^4/24 & b & b^2/2 & b^3/6 & b^4/24 \\ 0 & 1 & a & b & ab+c & bc & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2b & b^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & a & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & a & a^2/2 & a^3/6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & a & a^2/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & b & b^2/2 & b^3/6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & b & b^2/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & b \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The top-left  $6 \times 6$  block is the image of  $H$  in  $\mathrm{SL}(6, \mathbb{R})$ . The entries in  $A^n$  and  $B^n$  grow like  $n^2$ . This is beaten by the growth of some entries in the added blocks of size 5 which grow like  $n^4$ . This

gives rise to a representation of  $H$  of dimension  $6 + 4 + 4 = 14$ . The orbit of

$$[0 : 0 : 0 : 0 : 0 : 1 : 0 : 0 : 0 : 1 : 0 : 0 : 0 : 1]$$

is

$$[(a^4 + b^4)/24 + c^2 : bc : b^2 : c : b : 1 : a^3/6 : a^2/2 : a : 1 : b^3/6 : b^2/2 : b : 1]$$

so there is a codimension-4 projective hyperplane that is preserved, and which is defined by

$$x_6 = x_{10} = x_{14} \quad x_5 = x_{13} \quad x_3 = 2x_{12}$$

The restriction to this hyperplane gives  $\theta$ .

#### REFERENCES

- [1] D. Cooper, D. D. Long, and S. Tillmann. On convex projective manifolds and cusps. *Adv. Math.*, 277:181–251, 2015.

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